# Linear Algebra II

## 05/04/2016, Tuesday, 9:00 – 12:00

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You are **NOT** allowed to use any type of calculators.

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**1** $(3 + (3 + 6 + 3) = 15 \text{ pts})$ **Gram-Schmidt process**

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Consider the vector space $\mathbb{R}^5$ with the inner product $\langle x, y\rangle = x^T y$.

(a) Consider the vectors

$$x_1 = \begin{bmatrix}1 & 1 & 0 & 1 & 1\end{bmatrix}^T, \quad x_2 = \begin{bmatrix}1 & -1 & 1 & 0 & -1\end{bmatrix}^T, \quad x_3 = \begin{bmatrix}2 & 2 & 1 & -4 & 0\end{bmatrix}^T.$$

Let $\theta_{ij}$ denote the angle between $x_i$ and $x_j$. Find $\cos\theta_{12}$, $\cos\theta_{23}$, and $\cos\theta_{31}$.

(b) Let

$$A = \begin{bmatrix}1 & -1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 1\end{bmatrix}.$$

(i) Find a basis for $N(A)$, the null space of $A$.

(ii) Apply Gram-Schmidt process to obtain an orthonormal basis for $N(A)$.

(iii) Find the closest element of $N(A)$ to the vector $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$.

**2** $(9 + 6 = 15 \text{ pts})$ **Diagonalization**

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Consider the matrix

$$M = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a & b & c\end{bmatrix}$$

where $a$, $b$, and $c$ are real numbers.

(a) Determine all values of $(a, b, c)$ such that $M$ is unitarily diagonalizable.

(b) Find a unitary diagonalizer of $M$ for each triple $(a, b, c)$ found in (a).

**3** $(9 + 6 = 15 \text{ pts})$ **Eigenvalues/vectors and Cayley-Hamilton theorem**

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Let $A \in \mathbb{R}^{n\times n}$ be of the companion form

$$A = \begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1}\end{bmatrix}.$$

(a) Show that $A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I = 0$.

(b) Show that if $\lambda$ is an eigenvalue of $A$ then $\begin{bmatrix}1 & \lambda & \lambda^2 & \cdots & \lambda^{n-2} & \lambda^{n-1}\end{bmatrix}^T$ is an eigenvector corresponding to $\lambda$.

## 4 $(6+9=15$ pts) **Positive definiteness**

Consider the function

$$f(x,y,z) = ax^2 + y^2 + z^2 - xy + yz - xz$$

where $a \neq \frac{1}{3}$ is a real number.

(a) Find all stationary points of $f$.

(b) Determine all values of $a$ such that the stationary point(s) are local minimum.

## 5 $(10+5=15$ pts) **Singular value decomposition**

Let

$$M = \begin{bmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{bmatrix}$$

where $a$, $b$, and $c$ real numbers with $a > b > c$.

(a) Find a singular value decomposition of $M$.

(b) Find the best rank 2 approximation of $M$.

## 6 $(2+3+10=15$ pts) **Jordan canonical form**

Consider the matrix

$$M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) Find eigenvalues of $M$.

(b) Is $M$ diagonalizable? Why?

(c) Put $M$ into the Jordan canonical form.

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10 pts free